Inventory routing problems

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Transportation Center, Northwestern University, April 21st, 2009
The costs in a logistic network

- Fixed and variable production costs and inventory costs
- Fixed and variable transportation costs
- Inventory costs
Classical literature

**Production**: production planning, lot sizing, scheduling, ...

**Transportation**: fleet management, vehicle routing, ...

**Inventory**: inventory management.
Conflicting objectives

- Lot Size – Inventory
- Inventory – Transportation
- Lead Time – Transportation
- Product Variety – Inventory
- Cost – Customer Service
Conflicting objectives

- Lot Size – Inventory
- Inventory – Transportation
- Lead Time – Transportation
- Product Variety – Inventory
- Cost – Customer Service
If inventory costs are relevant...

- Reasons for low levels of inventory
  - reduce inventory costs
    (but then transportation is frequent and transportation costs are high)

- Reasons for rare transportation
  - reduce transportation costs
    (but then inventory levels and inventory costs are high)

Problem: Reduce inventory or transportation costs?

Answer: Min the sum of inventory and transportation costs
If inventory costs are not relevant...

- Reasons for rare transportation
  - reduce transportation costs
  (but then inventory levels may exceed capacity)

Min transportation costs subject to constraints on inventory levels
In both cases...

we have to work on time ....
...... and space
The literature

• Surveys
  Bertazzi, Savelsbergh, Speranza (2008), in ‘Vehicle routing’, Springer-Verlag
  Cordeau et al, in ‘Handbooks in Operations Research and Management Science: Transportation’, Forthcoming

• Pioneering papers
  Bell et al (1983), Interfaces
  Federgruen, Zipkin (1984), Operations Research
  Golden, Assad, Dahl (1984), Large Scale Systems
  Blumenfeld et al (1985), Transportation Research B
  ......
Transportation costs only: an example

The supplier has unlimited availability
Capacity of vehicles 5000

Initial inventory of \( i = \text{capacity} \)

Bell et al (1983), Interfaces
A natural solution

<table>
<thead>
<tr>
<th></th>
<th>capacity</th>
<th>consumption</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>5000</td>
<td>1000</td>
</tr>
<tr>
<td>B</td>
<td>3000</td>
<td>3000</td>
</tr>
<tr>
<td>C</td>
<td>2000</td>
<td>2000</td>
</tr>
<tr>
<td>D</td>
<td>4000</td>
<td>1500</td>
</tr>
</tbody>
</table>

Initial inventory of \( i \) = capacity
The supplier has unlimited availability
2 vehicles
Capacity of vehicles 5000

Repeat the red and the green tours every day
Daily transportation cost: 420
The optimal solution

<table>
<thead>
<tr>
<th></th>
<th>capacity</th>
<th>consumption</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>5000</td>
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<tr>
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<td>3000</td>
</tr>
<tr>
<td>C</td>
<td>2000</td>
<td>2000</td>
</tr>
<tr>
<td>D</td>
<td>4000</td>
<td>1500</td>
</tr>
</tbody>
</table>

Average daily transportation cost: 380
Inventory and transportation costs: an example

The single link problem in continuous time

Blumenfeld et al. (1985), Transportation Research B
The single link problem in continuous time

- No minimum inter-shipment time
- Single frequency $f$
- Continuous time between shipments $t = 1/f$
- Single vehicle

Optimal solution

$$t^* = \min \left( \sqrt{\frac{c}{hq}}, \frac{r}{vq} \right)$$

$$\begin{align*}
\min hqt + \frac{c}{t} \\
vqt \leq r \\
t \geq 0
\end{align*}$$
The single link problem in continuous time

transportation capacity \( r = 50 \)  
daily volume \( vq = 1 \times 5 = 5 \)  
fixed transportation cost \( c = 100 \)

Case 1: Shipped product is “cheap”  
daily unit inventory cost \( h = 1/30 \)

\[
t^* = \frac{r}{vq} = 10
\]

Case 2: Shipped product is “expensive”  
daily inventory cost \( h = 1/3 \)

\[
t^* = \sqrt{\frac{c}{hq}} = 7.745...
\]
The literature (deterministic)

- Deterministic product usage - inventory holding costs in the objective function
  Anily, Federgruen (1990), Management Sci.
  Speranza, Ukovich (1994), Operations Research
  ...... 

- Deterministic product usage - no inventory holding costs in the objective function
  Savelsbergh, Song (2006), Computers and Operations Research
  Gaur, Fisher (2004), Operations Research
  ......
The single link problem in discrete time

\[ t^* = \sqrt{\frac{c}{hq}} \]

non realistic

More complex models
with discrete time or shipping frequencies
The single link problem in discrete time

Discrete time: Frequency based policies

Discrete time: Any policy

Continuous time with minimum intershipment time

Models, algorithms and worst-case analysis of policies

Roundy (1985), MS
Speranza, Ukovich (1994), OR
Bertazzi, Speranza, Ukovich (2000), MS
Bertazzi, Speranza (2002), TS
Bertazzi, Chan, Speranza (2008), NRL

Example of worst-case result

\[
\frac{z^{policy}}{z^*} \leq 1.28
\]

min cost
The case of many customers

Models and solution approaches
- Archetti, Bertazzi, Hertz, Speranza (2009) submitted
- Archetti, Bertazzi, Laporte, Speranza (2008) TS
- Bertazzi, Paletta, Speranza (2002) TS
- Bertazzi (2008) MS

Analysis of direct shipping policies
- Bertazzi (2008) MS
- Gallego, Simchi-Levi (1990) MS and (1994) MS
- Hall (1985) MS

Asymptotic performance of policies
- Anily (1994) EJOR
- Chan, Federgruen, Simchi-Levy (1998) OR
- Chan et al (2002) MS
- Chan, Simchi-Levi (1998) MS
An inventory routing problem

How much to deliver to \( s \) at time \( t \) to minimize routing costs + inventory costs

- \( n \) customers
- \( H \) time horizon
- 1 vehicle

- \( r_{0t} \) Production at \( t \)
- \( B_t \) Inventory of 0 at \( t \)
- \( r_{st} \) Demand of \( s \) at \( t \)
- \( U_s \) Capacity of \( s \)
- \( I_{st} \) Inventory of \( s \) at \( t \)
Replenishment policies

Order-Up-to Level (OU)
Maximum Level (ML)

Different constraints on the quantities to deliver
The Order-Up-to Level policy

- Inventory at customer s
- Maximum level
- Minimum level
- Starting level
- Time

- $U_s$
- $L_s$
The Maximum Level policy

Every time a customer is visited, the shipping quantity is such that at most the maximum level is reached.
The OU and the ML policies: A comment

OU traditional, designed for a decentralized management

ML requires an integrated management
Decision variables

\( x_{st} \) : quantity \((\geq 0)\) shipped to customer \(s\) at time \(t\)

\[
z_{it} = \begin{cases} \frac{1}{2} & \text{if node } i \in M \ (\text{customer or supplier}) \\ 0 & \text{is visited at time } t \\ 0 & \text{otherwise} \end{cases}
\]

\[
y_{ij}^t = \begin{cases} 1 & \text{if the arc } (i,j) \text{ is traveled at time } t \\ 0 & \text{otherwise} \end{cases}
\]

\[
y_{i0}^t \in \{0,1,2\} \quad i \in M \quad t \in T
\]
The formulation

A mixed integer linear programming model

Min inventory and transportation costs

subject to

Inventory definition constraints
Stock-out constraints
Replenishment policy constraints
Capacity constraints
Routing constraints
## Exact methods and heuristics

### The Order-Up-to Level policy:

Local search  
Branch-and-cut (one vehicle)  
Archetti, Bertazzi, Hertz, Speranza (2008), *submitted*  
A hybrid heuristic (one vehicle)

### The Maximum Level policy:

Branch-and-cut (one vehicle)  
Archetti, Bertazzi, Hertz, Speranza (2008), *submitted*  
A hybrid heuristic (one vehicle)
A hybrid heuristic design

Exact approach allowed us to compute errors generated by the local search

Design of a tabu search

Design of a hybrid heuristic

Often large errors, rarely optimal

Sometimes large errors, sometimes optimal

Excellent results

Archetti, Bertazzi, Hertz, Speranza (2008), submitted
HAIR (Hybrid Algorithm for Inventory Routing)

- **Initialize** generates initial solution
- A **Tabu search** is run
  - Whenever a new best solution is found, **Improvements** is run
  - Every $JumpIter$ iterations without improvements, **Jump** is run
OU policy – Tabu search

**Search space**: feasible solutions
infeasible solutions
(violation of vehicle capacity or stock-out at the supplier)

**Solution value**: total cost + two penalty terms

**Moves for each customer**:
Removal of a day
Move of a day
Insertion of a day
Swap with another customer

**After the moves**:
Reduce infeasibility
Reduce costs
Route assignment

Objective: to find an optimal assignment of routes to days optimizing the quantities delivered at the same time. Removal of customers is allowed.

Optimal solution of a MILP model
The route assignment model

**Binary variables:**
- assignment of route r to time t
- removal of customer s from route r

**Continuous variables:**
- quantity to customer s at time t
- inventory level of customer s at time t
- inventory level of the supplier at time t
The route assignment model

Min inventory costs – saving for removals

s.t.

Stock-out constraints
OU policy defining constraints
Vehicle capacity constraints
Each route can be assigned to one day at most
Technical constraints on possibility to serve or remove a customer

# of binary variables: \((n+H) \times (\# \ of \ routes) + n \times H\)

NP-hard
OU policy – Improvements – MILP 1

Day 1

Day 2

Day 3

Day 4

Day 5

Day 6

Unused  Node removed  Incumbent solution

The optimal route assignment
Customer assignment

Objective: to improve the incumbent solution by merging a pair of consecutive routes. Removal of customers from routes, insertion of customers into routes and quantities delivered are optimized.

Optimal solution of a MILP model

For each merging and possible assignment day of the merged route a MILP is solved
The customer assignment model

**Binary variables:**
removal of customer $s$ from time $t$
insertion of customer $s$ into time $t$

**Continuous variables:**
quantity to customer $s$ at time $t$
inventory level of customer $s$ at time $t$
inventory level of the supplier at time $t$
The customer assignment model

Min inventory costs + insertion costs – saving for removals

s.t.

Stock-out constraints
OU policy defining constraints
Vehicle capacity constraints
Each route can be assigned to one day at most
Technical constraints on possibility to insert or remove a customer

# of binary variables: n*H

NP-hard
160 benchmark instances from Archetti et al (2007), TS

- $H = 3, n = 5, 10, \ldots, 50$
- $H = 6, n = 5, 10, \ldots, 30$
- Inventory costs low, high

Known optimal solution
# Summary of results

<table>
<thead>
<tr>
<th>Horizon</th>
<th>Policy</th>
<th>Inv. cost</th>
<th>% error</th>
<th>max % error</th>
<th>% error BPS</th>
<th>max % error BPS</th>
<th>number of optima</th>
</tr>
</thead>
<tbody>
<tr>
<td>H=3</td>
<td>OU</td>
<td>low</td>
<td>0.07</td>
<td>0.87</td>
<td>4.10</td>
<td>14.52</td>
<td>40/50</td>
</tr>
<tr>
<td></td>
<td></td>
<td>high</td>
<td>0.05</td>
<td>0.83</td>
<td>1.94</td>
<td>7.81</td>
<td>38/50</td>
</tr>
<tr>
<td>H=6</td>
<td>OU</td>
<td>low</td>
<td>0.14</td>
<td>1.84</td>
<td>3.51</td>
<td>9.50</td>
<td>21/30</td>
</tr>
<tr>
<td></td>
<td></td>
<td>high</td>
<td>0.08</td>
<td>0.55</td>
<td>1.70</td>
<td>5.01</td>
<td>18/30</td>
</tr>
<tr>
<td>H=3</td>
<td>ML</td>
<td>low</td>
<td>0.00</td>
<td>0.02</td>
<td>N.A.</td>
<td>N.A.</td>
<td>49/50</td>
</tr>
<tr>
<td></td>
<td></td>
<td>high</td>
<td>0.00</td>
<td>0.07</td>
<td>N.A.</td>
<td>N.A.</td>
<td>49/50</td>
</tr>
<tr>
<td>H=6</td>
<td>ML</td>
<td>low</td>
<td>0.12</td>
<td>1.17</td>
<td>N.A.</td>
<td>N.A.</td>
<td>12/30</td>
</tr>
<tr>
<td></td>
<td></td>
<td>high</td>
<td>0.17</td>
<td>0.76</td>
<td>N.A.</td>
<td>N.A.</td>
<td>11/30</td>
</tr>
</tbody>
</table>
Large instances

HAIR has been slightly changed:
- the improvement procedure is called only if at least 20 iterations were performed since its last application;
- the swap move is not considered.

Optimal solution unknown

- $H = 6$
- $n = 50, 100, 200$
- Inventory costs: low, high

10 instances for each size for a total of 60 instances
## Summary of results – large instances

<table>
<thead>
<tr>
<th>Horizon</th>
<th>Policy</th>
<th>Inv. cost</th>
<th>% error</th>
<th>max % error</th>
<th>% error BPS</th>
<th>max % error BPS</th>
</tr>
</thead>
<tbody>
<tr>
<td>H=6</td>
<td>OU</td>
<td>low</td>
<td>0.14</td>
<td>2.36</td>
<td>4.12</td>
<td>10.30</td>
</tr>
<tr>
<td></td>
<td></td>
<td>high</td>
<td>0.00</td>
<td>0.05</td>
<td>1.36</td>
<td>3.06</td>
</tr>
<tr>
<td>H=6</td>
<td>ML</td>
<td>low</td>
<td>0.03</td>
<td>0.40</td>
<td>N.A.</td>
<td>N.A.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>high</td>
<td>0.05</td>
<td>0.78</td>
<td>N.A.</td>
<td>N.A.</td>
</tr>
</tbody>
</table>

Running time for OU: 1 hour
Errors taken with respect to the best solution found within 5 min, 10 min, 30 min, 1 h

Running time for ML: 30 min
Errors taken with respect to the best solution found within 5 min, 10 min, 30 min

Running time BPS: always less than 3 min
Inventory-routing-production

Blumenfeld et al (1985) TR
Cohen and Lee (1988) OR
Chandra and Fisher (1994) EJOR
Thomas and Griffin (1996) EJOR

Bertazzi, Paletta, Speranza (2005), *J. Heur.*
Decomposition heuristic for OU

Archetti, Bertazzi, Paletta, Speranza (2008), submitted
Branch-and-cut for OU and ML and decomposition heuristic for ML
An inventory-routing-production problem

Production and shipping policies that globally optimize:

- Production planning
- Vehicle routing
- Inventory control

Bertazzi, Paletta, Speranza (2005), *J. Heur.*
RMI vs VMI

- **RMI: Retailer-Managed Inventory**
  
  Each retailer (customer) decides its replenishment policy.

- **VMI: Vendor-Managed Inventory**
  
  The supplier monitors the inventory level of each retailer (customer) and decides the replenishment policy of each retailer.
VMI vs RMI – the OU policy

A heuristic solution for the **RMI system** is obtained by serving each customer when its inventory level is down to 0 and then solving the distribution subproblem heuristically.

A heuristic solution for the **VMI system** is obtained by means of the decomposition of the problem in two subproblems: a production subproblem and a distribution subproblem.
## VMI vs RMI – the OU policy

<table>
<thead>
<tr>
<th>Condition</th>
<th>VMI Order-up</th>
</tr>
</thead>
<tbody>
<tr>
<td>general case</td>
<td>71%</td>
</tr>
<tr>
<td>no fixed transportation cost</td>
<td>92%</td>
</tr>
<tr>
<td>small production cost</td>
<td>50%</td>
</tr>
<tr>
<td>no retailer inventory cost</td>
<td>67%</td>
</tr>
</tbody>
</table>

Integrated modelling reduces costs

Cost of VMI with respect to RMI
VMI – the ML and the OU policy

A **branch-and-cut** algorithm for the OU and for the ML policy, for the case of **one vehicle** (MILP formulation similar to the inventory routing case)

A **heuristic** for the OU and for the ML policy, for the general case of **many vehicles**, based on the decomposition of the problem in two subproblems: a production subproblem and a distribution subproblem
Computational results

96 randomly generated problem instances

Four classes:

- 01-24: Basic
- 25-48: High production cost
- 49-72: High transportation cost
- 73-96: Zero inventory cost at customers
Small instances

One vehicle

19 customers, time horizon: 6

Optimal solution by the branch-and-cut algorithm implemented in C++ by using ILOG Concert 2 and CPLEX 9.0
The introduction of the production process makes the problem much more difficult.

- **Inventory + routing**
  
  \[ H=6 \quad n=30 \quad 2 \text{ hours}: \quad 99\% \text{ of the instances} \]

- **Production + inventory + routing**
  
  \[ H=6 \quad n=19 \quad 2 \text{ hours}: \quad 46\% \text{ of the instances} \]
Performance of the heuristic algorithm

<table>
<thead>
<tr>
<th>class</th>
<th>Average % error</th>
<th>Maximum % error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-24</td>
<td>1.35</td>
<td>3.31</td>
</tr>
<tr>
<td>25-48</td>
<td>0.19</td>
<td>0.42</td>
</tr>
<tr>
<td>49-72</td>
<td>1.77</td>
<td>5.05</td>
</tr>
<tr>
<td>73-96</td>
<td>0.33</td>
<td>1.18</td>
</tr>
</tbody>
</table>

Average error: 0.91%
Large instances

Fleet of vehicles

Comparison of OU with ML policies
### Order-up-to level vs Maximum Level

<table>
<thead>
<tr>
<th>VMI policy</th>
<th>50</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>ML</td>
<td>OU</td>
</tr>
<tr>
<td>Average error</td>
<td>0.02</td>
<td>5.92</td>
</tr>
<tr>
<td>Maximum error</td>
<td>1.04</td>
<td>33.67</td>
</tr>
<tr>
<td>Number of best solutions</td>
<td>91</td>
<td>5</td>
</tr>
<tr>
<td>Time</td>
<td>29.4</td>
<td>5.7</td>
</tr>
</tbody>
</table>

Maximum level policy is much better.
Conclusions

• The integrated models are very complex

• Advances in solution algorithms allow us to deal with integrated models

• Large savings can be obtained

Many research opportunities...