A Probit-based Time-Dependent Stochastic User Equilibrium Traffic Assignment Model

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The Transportation Center Seminar
Jan. 31, 2008
Outline

• Introduction
• Problem statement and assumptions
• TDSUE conditions and fixed-point formulation
• Temporal and spatial path correlation
• Two disaggregate micro-assignment implementation techniques
• Networks and numerical experiments
• Conclusions
Introduction

• DUE v.s. SUE
  – DUE does not have perception error to alternatives, SUE models a route choice behavior in an *imperfect information* network.

• Logit-based v.s. Probit-based
  – Some modified Logit-based models can represent alternative correlations, but Probit-based model allows a *general perception error term structure* by a well-specified variance-covariance matrix.

• Spatial path overlapping v.s. temporal-spatial path correlation
  – Spatial path overlapping only considers physical static correlation, the latter captures *dynamic and realistic path correlations*.

• Aggregate (flow-based macro-assignment) v.s. disaggregate (individual-based micro-assignment)
  – Disaggregate micro-assignment recognizes traveler behavior as an *individual choice* process, natural way to model *equilibrium with discrete choice* models.
Literature for Probit-based SUE

• Sheffi and Daganzo (1977)
  – Static, analytical travel cost function, Fixed-point formulation
  – simulation-based MSA link-based flow average method

• Maher et al. (1992, 1997, 2002)
  – Static, analytical travel cost function;
  – SAM, a Markovian stochastic network loading
  – Analytical approximation method to evaluate probit probabilities

• Cantarella and Cascetta (1995, 1997)
  – Probit-based SUE: Static with analytical travel function, simulation-based MSA link-based cost average method
  – Dynamic process assignment: find a stationary probability distribution state, Monte Carlo simulation of a stochastic process

  – probit-based SUE: Sensitivity analysis static problem
  – Stochastic process assignment: GSUE(2), simulation-based MSA link-based method with calculating shortest path for each individual
Problem statement and assumptions

• Assumptions
  – Given time-dependent OD demand tables and Time-varying network topology
  – All travelers with imperfect information and Random Utility Maximization (RUM) choice behavior
  – Perception error distribution across a choice set is Multivariate Normal (MVN)

• Solve for
  – Assignment of time-varying travelers/vehicles to a congested time-varying stochastic network under TDSUE conditions

• Methodology
  – Simulation-based MSA solution algorithm
  – Path generated and augmented to a grand path set at each iteration
  – A mesoscopic traffic simulator (DYNASMART) is used to obtain network cost and flow dynamics
TDSUE definition and Probit-based choice model

• TDSUE definition
  – For each OD pair and assignment/departure time interval, no user can reduce his/her perceived route travel cost/disutility by unilaterally changing routes.

• Probit-based choice model
  – Travel disutility
    \[ C = \bar{C} + \epsilon \quad \epsilon \sim MVN(0, \Sigma_{\epsilon}) \quad C \sim MVN(\bar{C}, \Sigma_{\epsilon}) \] (1)
  – Systematic utility function
    \[ V_i = -\theta \times \bar{C}_i \quad \forall i \in I \] (2)
  – Utility function
    \[ U = -\theta C = -\theta \bar{C} - \theta \epsilon = V + \xi \quad \xi \sim MVN(0, \Sigma_{\xi}) \quad U \sim MVN(V, \Sigma_{\xi}) \] (3)
TDSUE conditions and fixed-point formulation

• Path-based TDSUE conditions

\[ r_k^{\omega\tau} = q^{\omega\tau} \times p_k^{\omega\tau} [C_k^{\omega\tau}(r)] \quad \forall k \in K^\omega, \omega \in \Omega, \tau \in T \]  \hspace{0.5cm} (4)

• Path-based Fixed-point formulation

\[ r^* = q \times p[C(r^*)] \]  \hspace{0.5cm} (5)

• Gap function as a convergence criterion

\[ G(r) = \sum_{k \in K} \frac{1}{2} \times [q \times p_k(c_k(r)) - r_k]^2 \]  \hspace{0.5cm} (6)

\[ G(r_{n+1}) = \sum_{k \in K^{n+1}} \frac{1}{2} \times (q \times p_k^{n+1} - r_k^n)^2 \]  \hspace{0.5cm} (7)
Simulation-based MSA solution algorithm

**Step 1. Initial simulation-assignment.**
Set iteration counter \( n = 1 \). Perform a dynamic stochastic network loading and obtain the time-varying network link and path travel cost from simulator.

**Step 2. Time-dependent path generation.**
Solve the time-dependent shortest path problem to find new paths, and augment them into the existing grand path set \( \kappa^n \) for each OD pair and time interval.

**Step 3. Mean path cost and variance-covariance matrix calculation.**
Calculate the mean path travel cost and time-dependent variance-covariance matrix.

**Step 4. Route choice probability calculation.**
Explicitly calculate the auxiliary route choice probability, \( \bar{p}_k^n \) for \( \kappa^n \), by using the Monte Carlo simulation method based on the path cost and variance-covariance matrix from Step 3.

**Step 5. Dynamic micro-assignment:**
Based on the update of path assignment at Equation (14) and using MSA step size,

\[
\frac{1}{n+1} \times q^{n+1}
\]
portion of total number vehicles in the OD pair \( \omega \) and time interval \( \tau \) will have chance to re-select path based on the calculated auxiliary route choice probability at Step 4. A Monte Carlo draw is used to determine which vehicle will re-select path.

**Step 6. Update of path assignment and routing policy.**
Update the path assignment and routing policy by using the dynamic micro-assignment results from Step 5.

**Step 7. Dynamic stochastic network loading and simulation.**
Perform a dynamic stochastic network loading with updated path assignment and obtain the link and path cost from the simulator.

**Step 8. Convergence checking.**
Calculate the value of gap, \( G(r^{n+1}) = \sum_{k \in \kappa^{n+1}} \frac{1}{2} (y_k^{n+1} - r_k^n)^2 \), if \( \left| G(r^{n+1}) \right| < \phi \) or \( n = n_{\text{max}} \), then stop; otherwise, go to Step 2 and set \( n = n + 1 \).
Two methods of calculating temporal and spatial path correlation

- **I: Aggregated **overlapped** path travel time**

  \[
  \text{OverlappedTravelTime} = \max(0, \min(t_1 + TT_{ij}(t_1), t_2 + TT_{ij}(t_2)) - \max(t_1, t_2))
  \]

  \[
  \text{EmpiricalCov}(p1, p2) = \frac{1}{NOBS} \sum_{i=1}^{NOBS} (TT_i^{p1} - \bar{TT}^{p1}) \ast (TT_i^{p2} - \bar{TT}^{p2})
  \]
An example: three-route network

- Three-route network:

- V-C matrix at time interval 40
  - Aggregated path overlapping
  - Empirical statistics correlation


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<thead>
<tr>
<th>Variance-Covariance</th>
<th>Temporal and spatial path overlapping</th>
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<tbody>
<tr>
<td></td>
<td>Path A</td>
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<td>Path A</td>
<td>1.66627</td>
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<tr>
<td>Path B</td>
<td>0.00000</td>
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<tr>
<td>Path C</td>
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</table>

<table>
<thead>
<tr>
<th>Variance-Covariance</th>
<th>Empirical statistics correlation</th>
</tr>
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<tbody>
<tr>
<td></td>
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<td>Path A</td>
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<tr>
<td>Path B</td>
<td>-0.00005</td>
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<tr>
<td>Path C</td>
<td>-0.00009</td>
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</table>
Probit-based choice probability calculation

- **Probability function:**
  \[
p_i = \int_{u_1=-\infty}^{u_i} \int_{u_2=-\infty}^{u_i} \cdots \int_{u_i=-\infty}^{u_i} \left[ (2\pi)^J |\Sigma| \right]^{-\frac{1}{2}} \exp \left[ -\frac{1}{2} (U-V)^T \Sigma^{-1} (U-V) \right] du_1 du_2 \cdots du_i \cdots du_j
\]  
  
- **Evaluation methods**
  - Numerical Integration
    - Small number of choice set (<= 3)
  - Analytical approximation
    - Not accurate
  - **Monte Carlo simulation**
    - Good accuracy and can be adapted to a disaggregate micro-assignment framework efficiently.
Path assignment update: aggregate v.s. disaggregate

- flow update: aggregate macro-assignment

\[ r_{k}^{n+1} = r_{k}^{n} + \lambda^{n} \times (y_{k}^{n} - r_{k}^{n}) \]  \hspace{1cm} (11)

- Routing policy update: disaggregate micro-assignment

\[ r_{k}^{n+1} = r_{k}^{n} + \lambda^{n} \times (y_{k}^{n} - r_{k}^{n}) \]
\[ = r_{k}^{n} + \lambda^{n} \times (q_{\omega}^{\tau} \times \bar{p}_{k}^{n} - r_{k}^{n}) \]
\[ = q_{\omega}^{\tau} \times p_{k}^{n} + \lambda^{n} \times (q_{\omega}^{\tau} \times \bar{p}_{k}^{n} - q_{\omega}^{\tau} \times p_{k}^{n}) \]
\[ = (1 - \lambda^{n}) \times q_{\omega}^{\tau} \times p_{k}^{n} + \lambda^{n} \times q_{\omega}^{\tau} \times \bar{p}_{k}^{n} \]  \hspace{1cm} (12)

- \( \lambda^{n} \times q_{\omega}^{\tau} \) vehicles will have chance to re-choose path based on auxiliary choice probability.
Two disaggregate micro-assignment implementation techniques

• Explicit approach
  – Stable and accurate probability by Monte Carlo simulation technique
  – A portion of vehicles re-choose path based on the auxiliary choice probability

• Implicit approach
  – Each one of a portion of vehicles that needs to re-choose path will choose the best path from a given path set based on a single Monte Carlo random draw.
  – Auxiliary choice probability, routing policy, and path assignment will be obtained after micro-assignment of each vehicle
Flow chart of two disaggregate implementation techniques

1. **Explicit Approach**
   1. Initial simulation-assignment
   2. Time-dependent shortest path calculation
   3. Mean path travel cost and variance-covariance matrix calculation
   4. Route probability calculation: Monte Carlo simulation
   5. Dynamic micro-assignment
   6. Update of path assignment and routing policy
   7. Dynamic stochastic network loading and simulation
   8. Convergence checking

   If No, repeat from 1.
   If Yes, stop.

2. **Implicit Approach**
   1. Initial simulation-assignment
   2. Time-dependent shortest path calculation
   3. Mean path travel cost and variance-covariance matrix calculation
   4. Dynamic micro-assignment
   5. Update of path assignment, routing policy, and auxiliary probability
   6. Dynamic stochastic network loading and simulation
   7. Convergence checking

   If No, repeat from 5.
   If Yes, stop.
Two practical large-scale networks

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<thead>
<tr>
<th>Network</th>
<th># of zones</th>
<th># of nodes</th>
<th># of links</th>
<th>Planning horizon</th>
<th>Assignment interval</th>
<th># of vehicles</th>
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<tbody>
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<td>180</td>
<td>445</td>
<td>2 hours</td>
<td>5 min</td>
<td>37858</td>
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<tr>
<td>CHART, MD</td>
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<td>3387</td>
<td>2 hours</td>
<td>5 min</td>
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Fort Worth, TX

Baltimore-Washington CHART, MD
Numerical results (1)
--convergence pattern

Convergence pattern for Fort Worth network

Convergence pattern for CHART network
### Numerical results (2)

--- Computational time

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<th>Iteration</th>
<th>Fort Worth Network</th>
<th>CHART Network</th>
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<td>Average</td>
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Conclusions

• **Probit-based TDSUE** can model route choice *behavior realistically* in an *imperfect* information dynamic transportation network.

• **Fixed-point formulation** of TDSUE problem could readily accommodate a *richer set of behavioral dimensions* beyond simply route choice.

• **Two temporal and spatial path correlation** calculation methods are proposed: *aggregated overlapped* path travel time, and *sampled vehicle experienced* path travel time empirical statistics; the latter can represent both *overlapping* and *non-overlapping* correlations.

• **Two disaggregate micro-assignment implementation techniques** for *large-scale* networks are presented: *explicit* probability calculation, and *implicit* sampling micro-assignment without Probit probability evaluation; the latter is more *efficient* in terms of computational time with *good* convergence pattern.
Thank you & comments!
Path overlapping

• Spatial overlapping: (Modified Logit and static Probit-based models)

\[
\text{Cov}(p_1, p_2) = \sum_{(i,j) \in A(p_1) \cup A(p_2)} TT_{ij} \times \delta_{p_1,p_2}^i \delta_{p_1,p_2}^j = \{0,1\}
\]

• Temporal and spatial overlapping

- **t1**: Arrival time at upstream node i of path p1
- **t2**: Arrival time at upstream node i of path p2
- **t1 + TT(t1)**: Arrival time at downstream node j of path p1
- **t2 + TT(t2)**: Arrival time at downstream node j of path p2