Outline

• Motivation
• A Dynamic Routing Problem
• Illustration
• Dispatch Policies
• Analysis
• Computational experiments
• A Stochastic & Dynamic Routing Problem
Dynamic Routing

• Dynamic Routing
  – Information about customers to visit is not revealed at once, but over time
  – Adjust vehicle schedules as new information becomes available

• Growing Importance
  – Availability of exact position of vehicles at every moment in time (GPS)
  – Capability to communicate with vehicles at all times (wireless)
Literature (1)

• Dynamic Routing
  – Adjust routes as new information becomes available
  • Psaraftis (1988)
  • Gendreau, Guertin, Potvin, Taillard (1999)
  • Ichoua, Gendreau, Potvin (2000)
  • Larson, Madsen, Solomon (2002)
  • Ghiani, Guerriero, Laporte, Musmanno (2003)

No assumptions about future demands
• Stochastic Routing
  – Determine an a priori routing plan minimizing expected costs
    • Dror, Laporte, Trudeau (1989)
    • Secomandi (2001)
    • Laporte, Louveau, Van Hamme (2002)

• Dynamic and Stochastic Routing
  • Bertsimas, Van Ryzin (1991)
  • Bertsimas, Simchi-Levi (1996)
  • Bent, Van Hentenryck (2004)
A Dynamic Routing Problem

- Motivating Real-Life Application
  - Service contracts
    - A guarantee that service takes place within a specified number of days of the receipt of the service request
  - Distribution planning
    - Each day, decide which service requests to fulfill and how based on partial information (no knowledge of future service requests)
Daily Dispatching

• Each day there are two types of service requests:
  – Those that have to be fulfilled today
  – Those that can be fulfilled today, but may be postponed to the next day

• No knowledge of the service requests that will arrive tomorrow (or further into the future)

Question: Which, if any, of the customers that can be fulfilled today, but may be postponed to the next day should be fulfilled today?
Illustration (Traditional Routing)
Illustration (Traditional Routing)
Illustration (Dynamic Routing)

- Today
- Today or tomorrow
Illustration (Dynamic Routing)

Option 1

- **Today**
- **Today or tomorrow**
Illustration (Dynamic Routing)

Option 2

- Today
- Today or tomorrow
Illustration (Dynamic Routing)

Option 3

- Today
- Today or tomorrow
Illustration (Dynamic Routing)

Option 4

- **Today**
- **Today or tomorrow**
Illustration (Dynamic Routing)

- Realization 1

- Should we have postponed?

- Today
- Today or tomorrow
- Tomorrow or the day after

- Should we have postponed?
Illustration (Dynamic Routing)

Realization 2

- Today
- Today or tomorrow
- Tomorrow or the day after

Should we have postponed?
Illustration (Dynamic Routing)

To Serve or Not to Serve: That is the Question

Should we postpone?

Today

Today or tomorrow
Problem Definition

- Time periods (days): \( t = 0, 1, 2, 3, \ldots, T \)
- Orders \( C_{t|t+1} \)
  - arrive at the beginning of day \( t \).
  - have to be fulfilled on day \( t \) or day \( t+1 \)
- Single vehicle with infinite capacity
- Minimize the average daily costs

Example \( T = 4 \)
Daily decision problem

• On day t:
  – Set $\hat{C}_{t-1|t} \subseteq C_{t-1|t}$ of orders to be served on day t (postponed in t-1)
  – Set $C_{t|t+1}$ of orders to be served on day t or day t+1

• Find a **dispatch policy** to answer the following question:
  – Which subset of the orders in $C_{t|t+1}$ should be served on day t?

• Observe:
  – Orders $C_{0|1}$ are always postponed to day 1
  – Orders $C_{T|T+1}$ are always served on day T
  – T-1 actual decisions

Example $T = 4$
A Dynamic Routing Problem

- Simple Dispatch Policies
  - IMMEDIATE:
    - Each day serve new customers immediately
  - DELAY:
    - Each day delay serving new customers
  - SMART:
    - Each day decide whether to serve new customers immediately or to delay serving new customers

*Note: We do not consider serving a subset of newly arrived customers immediately and postponing the service of the remaining customers*
Competitive Ratio

\[ z(I, A) := \text{Value of Algorithm A on instance I} \]
\[ z^*(I) := \text{Optimal off-line value for instance I} \]

Note: off-line optimization problem is NP-hard

Competitive ratio

\[ r_A := \max_I \frac{z(A, I)}{z^*(I)} \]

Note: competitive ratio tries to capture the value of information
A simple case

- Non-negative Real Line (\( R^+ \))

Service schedule

- **Today**
- **Today or tomorrow**

Only important customers
Special Case – Two Periods

• Customer sets
  – \( C_1, C_{1|2}, C_2 \)

• Tour lengths
  – \( L_1, L_{1|2}, L_2 \)
  – \( L_{1,1|2}, L_{1|2,2}, L_{all} \)

• Triangle inequality
  – \( \text{Max}(L_1, L_{1|2}) \leq L_{1,1|2} \leq L_1 + L_{1|2} \)
  – \( \text{Max}(L_{1|2}, L_2) \leq L_{1|2,2} \leq L_{1|2} + L_2 \)

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simplified notation

\[
\begin{align*}
C_1 &= C_{0|1} \\
C_2 &= C_{2|3}
\end{align*}
\]
Theorem The competitive ratio of any algorithm with customer locations on the non-negative real line is greater than or equal to $\sqrt{2}$.

$$r_A := \max_I \frac{z(A, I)}{z^*(I)}$$
Case 1: Algorithms visiting $C_1$ and $C_{1|2}$ customers

$$r_{A'} = \frac{z(A', I)}{z^*(I)} = \frac{2a + 2a}{2 + 2a} = \frac{2a}{1 + a}$$

Case 2: Algorithms visiting $C_1$ customers

$$r_{A''} = \frac{z(A'', I)}{z^*(I)} = \frac{2 + 2a}{2a} = \frac{1 + a}{a}$$
Real Line

\[ r_A := \max_I \frac{z(A, I)}{z^*(I)} \]

\[ r_A \geq \min\left(\frac{2a}{1+a}, \frac{1+a}{a}\right) \]

\[ \max_{a>1}\{\min_{a>1}\left(\frac{2a}{1+a}, \frac{1+a}{a}\right)\} = \sqrt{2} \]
Algorithm IMMEDIATE: Visit all customers in $C_1$ and $C_{1|2}$ in the first time period and visit all customers in $C_2$ in the second time period.

Theorem The competitive ratio of Algorithm IMMEDIATE with customer locations on the non-negative real line is 2.
Theorem  The competitive ratio of Algorithm IMMEDIATE with customer locations on the non-negative real line is 2.

\[ r_A := \max_I \frac{z(A, I)}{z^*(I)} \]

\[ z(IMMEDIATE, I) = L_{1,1} + L_2 \]

\[ z^*(I) \geq L_{all} \]

\[ r_{IMMEDIATE}(I) \leq \frac{L_{1,1} + L_2}{L_{all}} \leq \frac{2L_{all}}{L_{all}} = 2 \]
Algorithms

**Theorem** The competitive ratio of Algorithm IMMEDIATE with customer locations on the non-negative real line is 2.
Algorithm **DELAY**: Visit all customers in $C_1$ in the first time period and visit all customers in $C_{1|2}$ and $C_2$ in the second time period.

**Theorem** The competitive ratio of Algorithm DELAY with customer locations on the non-negative real line is 2.
Theorem  The competitive ratio of Algorithm DELAY with customer locations on the non-negative real line is 2.

\[ r_A := \max_I \frac{z(A,I)}{z^*(I)} \]

\[ z(DELAY, I) = L_1 + L_1|2,2 \]

\[ z^*(I) \geq L_{all} \]

\[ r_{IMMEDIATE}(I) \leq \frac{L_1 + L_1|2,2}{L_{all}} \leq \frac{2L_{all}}{L_{all}} = 2 \]
Theorem  The competitive ratio of Algorithm DELAY with customer locations on the non-negative real line is 2.
Algorithm \textsc{SMART}(p): If $L_{1,1|2} \leq pL_1$, then apply \textit{IMMEDIATE}, else apply \textit{DELAY}.

\textbf{Theorem} The competitive ratio of Algorithm \textsc{SMART}(1 + \sqrt{2}) for instances with customer locations on the non-negative real line is $\sqrt{2}$.

\textbf{Best Possible Algorithm}
Algorithms

**Theorem** The competitive ratio of Algorithm \( SMART(1 + \sqrt{2}) \) for instances with customer locations on the non-negative real line is \( \sqrt{2} \).
**Theorem** The competitive ratio of any algorithm for a planning horizon $T > 2$ and with customer locations on the non-negative real line is greater than or equal to 1.44.
Case 1: Algorithms visiting $C_1$ and $C_{1|2}$

\[ r_{A'} = \frac{z(A', I)}{z^*(I)} = \frac{2a + 2a}{2 + 2a} = \frac{2a}{1 + a} \]

Case 2a: Algorithms visiting $C_1$ + visiting $C_{1|2}$ and $C_{2|3}$

\[ r_{A''} = \frac{z(A'', I)}{z^*(I)} = \frac{2 + 2ab + 2ab}{2a + 2ab} = \frac{1 + 2ab}{a + ab} \]

Case 2b: Algorithms visiting $C_1$ + visiting $C_{1|2}$

\[ r_{A'''} = \frac{z(A''', I)}{z^*(I)} = \frac{2 + 2a + 2ab}{2 + 2ab} = \frac{1 + a + ab}{1 + ab} \]
Real Line

\[ r_A \geq \min \left( \frac{2a}{1 + a}, \frac{1 + 2ab}{a + ab}, \frac{1 + a + ab}{1 + ab} \right) \]

\[ r_A \geq \min \left( \frac{2a}{1 + a}, \frac{2a - 1 + 2\sqrt{a(2a - 1)}}{2a - 1 + \sqrt{a(2a - 1)}} \right) \]

decreases with \( b \) increases with \( b \)

decreases with \( a \)

increases with \( a \)
Theorem The competitive ratio of Algorithm IMMEDIATE with customer locations on the non-negative real line is 2 for any length of the planning horizon.

\[ z(IMMEDIATE, I) = L_{1,1|2} + L_{2|3} + \ldots + L_{T-1|T} + L_T. \]

\[ z^*(I) \geq L_{1|2} + L_{3|4} + \ldots + L_{T-1|T} \]

\[ z^*(I) \geq L_1 + L_{2|3} + \ldots + L_{T-2|T-1} + L_T \]

\[ 2z^*(I) \geq L_{1,1|2} + L_{2|3} + \ldots + L_{T-2|T-1} + L_{T-1|T} + L_T \]
Theorem The competitive ratio of Algorithm DELAY with customer locations on the non-negative real line is 2 for any length of the planning horizon.
Theorem The competitive ratio of Algorithm SMART\((p)\) with customer locations on the non-negative real line and \(T = 3\) is \(\max\left(\frac{1+2p^2}{p+p^2}, \frac{1+p}{p}\right)\) with a best ratio of \(\frac{3}{2}\) for \(p = 2\).
Algorithms

**Time-Dependent** $SMART(p_1, p_2)$: Apply $SMART(p_1)$ in the first time period and $SMART(p_2)$ in the second time period.

**Theorem**: The competitive ratio of Algorithm $SMART(p_1, p_2)$ with customers on the non-negative real line is

$$\max\left(\frac{2p_1}{1 + p_1}, \frac{1 + p_1}{p_1}, \frac{1 + p_2}{p_2}, \frac{1 + 2p_1p_2}{p_1 + p_1p_2}\right),$$

with a best ratio of 1.473 for $p_1 \approx 2.79$ and $p_2 \approx 2.11$.

**SMART(2) is not best possible**
The competitive ratio of Algorithm SMART (2) with customer locations on the non-negative real line is \( \frac{3}{2} \) for any length of the planning horizon.

For any instance \( C \) with planning horizon \( T \), either:

- An instance \( C' \) exists with shorter planning horizon such that
- We can directly prove

\[
\frac{z(\text{SMART}, C)}{z^*(C')} \leq \frac{3}{2}
\]

\[
\frac{z(\text{SMART}, C)}{z^*(C)} \leq \frac{3}{2}
\]
Structure of optimal solutions

• Every solution can be represented by a sequence of $T-1$ decisions of the type $I$ or $D$

• Substitution rules
  - $...II... \rightarrow ...ID...$
  - $...DD... \rightarrow ...ID...$

• An optimal solution can be represented by a sequence of the type:
  - $.........IDIDIDIDIDID.........$ or
  - $.........IDIDI\textcolor{red}{X}DIDIDID.........$
Worse instance with shorter time horizon (1)

\[
\frac{z(SMART, C)}{z^*(C)} \rightarrow \ldots IDDID\ldots \quad \text{or} \quad \ldots DIDD\ldots
\]

\[\begin{array}{cccccc}
C, T=6 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
C_01 & C_12 & C_23 & C_34 & C_45 & C_56 & C_67 \\
C', T'=3 & 0 & 1 & 2 & 3 & \{\emptyset\} & \\
C_01 & C_12 & C_23 & \\
C'', T''=3 & 3 & 4 & 5 & 6 & \\
C_34 & C_45 & C_56 & C_67 \\
\end{array}\]

\[
z(SMART, C) = \frac{z(SMART, C') + z(SMART, C'')}{z^*(C')} \leq \max \left( \frac{z(SMART, C')}{{z^*(C')}, \frac{z(SMART, C'')}{z^*(C'')}} \right)
\]
Worse instance with shorter time horizon (2)

\[
\frac{z(\text{SMART}, C)}{z^*(C)} \rightarrow \ldots \text{DIDID} \quad \ldots \text{IDIDI}
\]

\[ C, \; T=6 \]

\[
\begin{align*}
C_0|1 & \quad C_1|2 & \quad C_2|3 & \quad C_3|4 & \quad C_4|5 & \quad C_5|6 & \quad C_6|7 \\
0 & \quad 1 & \quad 2 & \quad 3 & \quad 4 & \quad 5 & \quad 6 
\end{align*}
\]

\[ C', \; T'=5 \]

\[
\begin{align*}
C_0|1 & \quad C_1|2 & \quad C_2|3 & \quad C_3|4 & \quad C_4|5 & \quad C_5|6 \\
0 & \quad 1 & \quad 2 & \quad 3 & \quad 4 & \quad 5 & \quad 6 
\end{align*}
\]

\[
\frac{z(\text{SMART}, C)}{z^*(C)} \leq \frac{z(\text{SMART}, C') + L_{T|T+1}}{z^*(C')} \leq \frac{z(\text{SMART}, C')}{z^*(C')}
\]
Direct proof of the bound (case T odd)

\[
\frac{z(\text{SMART}, C)}{z^*(C)} \rightarrow \begin{array}{cccccc}
\text{DI...DIDI} \\
\text{ID...IDID}
\end{array}
\]

\[
z(\text{SMART}, C) = L_{0|1} + \max(L_{1|2}, L_{2|3}) + \max(L_{3|4}, L_{4|5}) + L_{5|6}
\]

\[
z^*(C) = \max(L_{0|1}, L_{1|2}) + \max(L_{2|3}, L_{3|4}) + \max(L_{4|5}, L_{5|6})
\]
If the Theorem is not true, the system of equations below should have a feasible solution

\[ z(\text{SMART}) \geq \left( \frac{3}{2} + \varepsilon \right) z^* \]

\[ z(\text{SMART}) = L_{0|1} + \sum_{0< t \leq T \atop t \text{ even}} \max(L_{t-1|t}, L_{t|t+1}) + L_{T|T+1} \]

\[ z^* = \sum_{0< t \leq T \atop t \text{ odd}} \max(L_{t-1|t}, L_{t|t+1}) \]

\[ L_{1|2} \geq pL_{0|1} \]

\[ L_{t|t+1} \leq pL_{t-1|t} \quad \text{for } t > 0, \text{ even} \]

\[ L_{t|t+1} \geq 0 \quad \text{for } t = 0,1,\ldots,T \]

\[ M_t = \max(L_{t-1|t}, L_{t|t+1}) \]

\[ M_t \geq L_{t-1|t} \]

\[ M_t \geq L_{t|t+1} \]

\[ M_t \leq L_{t-1|t} + M \cdot (1 - x_t) \]

\[ M_t \leq L_{t|t+1} + M \cdot x_t \]

\[ M_t \geq 0 \]

\[ x_t \in \{0,1\} \]
• Farkas Lemma

\[
\begin{aligned}
A_\varepsilon y &= b \\
y &\geq 0
\end{aligned}
\quad \text{is infeasible} \quad \Leftrightarrow \quad \begin{aligned}
vA_\varepsilon &\geq 0 \\
v b &< 0
\end{aligned}
\quad \text{is feasible}
\]

\[
b = b\left([x_1 \cdots x_T]^t\right)
\]
\[
y = \left[L_{0|1} \cdots L_{T|T+1}\right]^t
\]

Issue: Farkas Lemma applies to linear systems with continuous variables. Our system has integer variables.
Euclidean Plane

**Theorem** The competitive ratio of any algorithm with customer locations in the Euclidean plane is greater than or equal to $\frac{3}{2}$
Choice 1

\[ r_A = \frac{1+(2+\epsilon)}{2+3\epsilon} \]

- **Today**
- **Today or tomorrow**
- **Tomorrow or the day after**
Choice 2

\[ r_A = \frac{(2+\epsilon)+1}{2+\epsilon} \]

- Today
- Today or tomorrow
- Tomorrow or the day after
Choice 3

\[ r_A = \frac{(1 + \epsilon) + (2 + \epsilon)}{(1 + 2\epsilon) + (1 + \epsilon)} \]

- Today
- Today or tomorrow
- Tomorrow or the day after
Euclidean plane

• What is the competitive ratio of SMART(p)?

• Is it still a good idea to apply either IMMEDIATE or DELAY?

• If not, how to partitioning the set of newly arrived customers?
Which orders should be included in today's tour?
Implementation of SMART(p)

Idea:

- $X = C_{t|t+1}$
- $Y = \hat{C}_{t-1|t}$
- Add orders from $X$ to $Y$ as long as $TSP(Y) \leq p TSP(\hat{C}_{t-1|t})$

Which order to be selected?

- For each $v \in X$
  - $D_v = \text{Saving resulting from extraction from } TSP(X)$
  - $I_v = \text{Insertion cost in } TSP(Y)$
- Select $v$ which maximizes $D_v - I_v$
Computational results

• Problem settings:
  – Orders uniformly distributed in a square
  – Orders non-uniformly distributed in a square
Uniform Instances

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Non-Uniform Instances
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**Non-Uniform**
Perfect Information

Single customer per period

Cost of serving customer arriving t+1 by itself

Cost of serving customer arriving t by itself

Cost of serving customer arriving in t and the customer arriving in t+1 together

Immediate

Delay
Perfect Information

Two customers per period

Immediate, Immediate

Immediate, Delay

Delay, Immediate

Delay, Delay

Optimal solution with perfect information can be computed when the number of customers arriving per period is small.
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Value of Information?

- Can we do better if we know the distribution of the number and location of future requests?
- What will policies look like in that case?

A stochastic & dynamic routing problem
Real Line Setting

\[ M \left\{ \begin{array}{l}
\text{for} \theta \in \mathbb{D}, b \in \mathbb{P}, \text{if } \theta \in \mathbb{R} \cup \mathbb{X} \end{array} \right. \]

\[ f(x, y, a) := \begin{cases}
0 & \text{if } a = 0 \\
\max \{x, y\} & \text{if } a = 1
\end{cases} \]

\[ g(x, y, a) := \begin{cases}
0 & \text{if } a = 0 \\
\max \{x, y\} & \text{if } a = 1
\end{cases} \]

\[ x_t \in [0, 1] := \text{location of delayed customer (if any)} \]
\[ y_t \in [0, 1] := \text{location of newly arrived customer} \]

Assume \( x_0 \) is given and \( \{y_t\}_{t=0}^{\infty} \) is an independent and identically distributed sequence with common distribution function \( F \).
Problem:

Optimal value function:

Optimality equation:
There is a threshold function $z^*$ : $[0,1] \mapsto [0,1]$ such that it is optimal to visit the customer whose request is received at time $t$ if $t \leq z^*(x_t)$, and it is optimal not to visit the customer whose request is received at time $t$ if $t > z^*(x_t)$. 

...
Expected Total Discounted Cost

\[ b^{\cdots}, \langle \mathcal{D} r \rangle \rightarrow M \delta \]

\[ S \quad Y^{\rangle} \rightarrow M \Delta = M \]

\[ S \quad Y^{\rangle} \rightarrow f d I \rightarrow f \]

\[ S \quad Y^{\rangle} \rightarrow H d I \rightarrow H \]

\[ S \quad Y D, \ldots, \bullet r^r r \bullet <, \bullet < \bullet < \bullet < f z, \mathcal{D} Y \rightarrow M d I \rightarrow f 5 \]

\[ \bullet, i, \ldots, X \bullet D, r \, \bullet^b, r \bullet<, \bullet^p \bullet, <y< \]

\[ S \quad \Phi H \cdot M_f \cdot M_Q \cdot f 5 <y^\rangle, H \cdot Y \rangle M_d t Y \cdot M_d \cdot M_Q t M_f \]
Approximating Optimal Value Function

\[ \hat{T}(V)(x) := \int_{[0,1]} \min\{x + \alpha V(y), \max\{x, y\} + \alpha V(0)\} \, d\hat{F}(y) \]

\[ T^*(V)(x) := \int_{[0,1]} \min\{x + \alpha V(y), \max\{x, y\} + \alpha V(0)\} \, dF(y) \]

\[ \text{Mapping} \: T : V \mapsto V \]
Approximating Optimal Value Function

\[ \hat{\pi}(x, y) := \arg \max_{a \in \mathcal{A}} \{ g(x, y, a) + \alpha \hat{T}_{i+1}(V)(f(x, y, a)) \} = \begin{cases} 1 & \text{if } f(x, y) \geq \max\{x, y\} + \alpha \hat{T}_{i+1}(V)(0) \\ 0 & \text{otherwise} \end{cases} \]

\[ \| V^* - \hat{V}_{\pi} \| \leq 2\epsilon + \alpha^2 \theta \]

\[ \forall V \forall i : \| T^*_i(\hat{T}_i(V)) - \hat{T}_i(V) \| \leq \epsilon \]

\[ \exists V \exists i : \| \hat{T}_i(V) - \hat{T}_{i-1}(V) \| \leq \theta \]
Approximating Optimal Value Function

\[ f: [0,1] \mapsto \mathbb{R} \]

Approximating \( \int_{[0,1]} f(y) dF(y) \) by discretizing support \([0,1]\) of \( F \)

Probability distribution \( F \) on \([0,1]\)

\[ y_m, j \in \left( \frac{j-1}{m}, \frac{j}{m} \right] \text{ for } j = 1, \ldots, m \]

\[ f_m(0) := f(y_{m,1}), y \in \left( \frac{j-1}{m}, \frac{j}{m} \right], f_m(y) := f(y_{m,j}) \]

\[ \int_{[0,1]} f_m(y) dF(y) = \int_{[0,1]} f(y) dF_m(y) \]

\[ F_m(y) := F\left( \frac{1}{m} \right) I\{y \geq y_{m,1}\} + \sum_{j=2}^{m} \left[ F\left( \frac{j}{m} \right) - F\left( \frac{j-1}{m} \right) \right] I\{y \geq y_{m,j}\} \]
Uniform Distribution

Single customer arriving each period
Uniform Distribution

\[ f(x,y) = \begin{cases} 1 & \text{if } (x,y) \in [0,1] \times [0,1] \\ 0 & \text{o.w.} \end{cases} \]

Expected distance from \( c \) to 0

\[ E[c_0] = \int_{-1}^{1} \int_{-1}^{1} \sqrt{x^2 + y^2} f(x,y) \, dy \, dx = 0.765195 \]
Uniform Distribution

\[ E[bc] = \int_{-1}^{1} \int_{-1}^{1} \sqrt{(x-x_b)^2 + (y-y_b)^2} f(x,y) \, dy \, dx \]
Policy: If $\mathcal{A}h^1 \mathcal{T} \mathcal{H}_r \bullet a^1 \bullet b$, then serve $b$ immediately else delay serving $b$. 

$\mathbb{E}[bc] - \mathbb{E}[c_0] > ab - ao \quad a \in \hat{C}_{t-1} \mid t \quad b \in C_t \mid t+1$
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Uniform
Questions

Comments